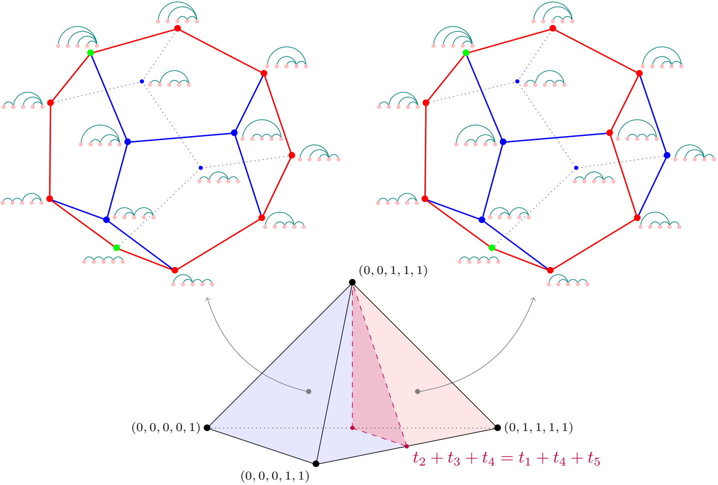
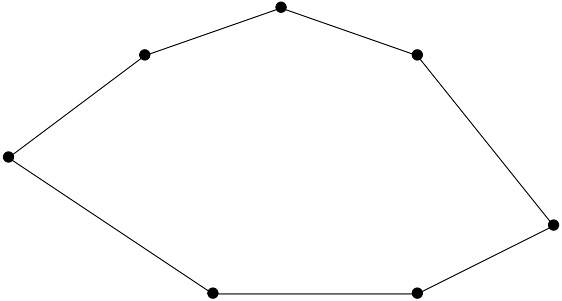
Cyclic Associahedra and intrinsic degrees

Aenne Benjes, **Germain Poullot** & Raman Sanyal



Pivot rules and projections of associahedra

Optimization in dimension 2:



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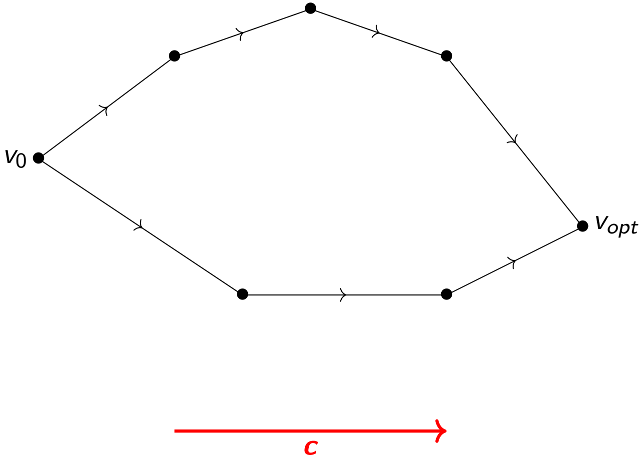

Shadow vertex rule

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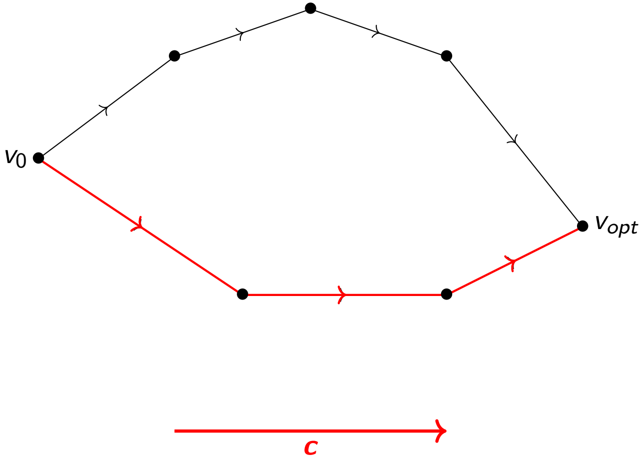

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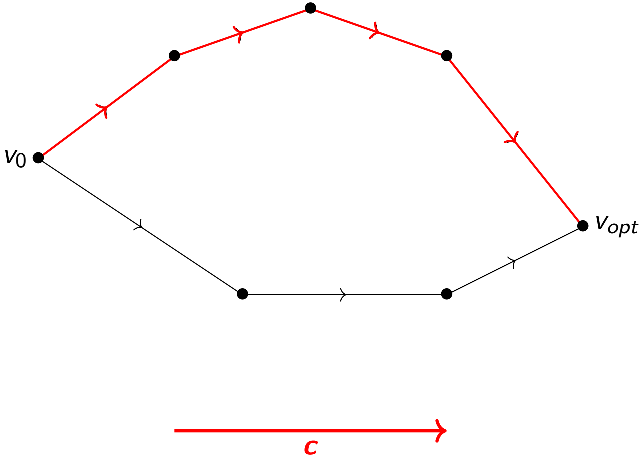
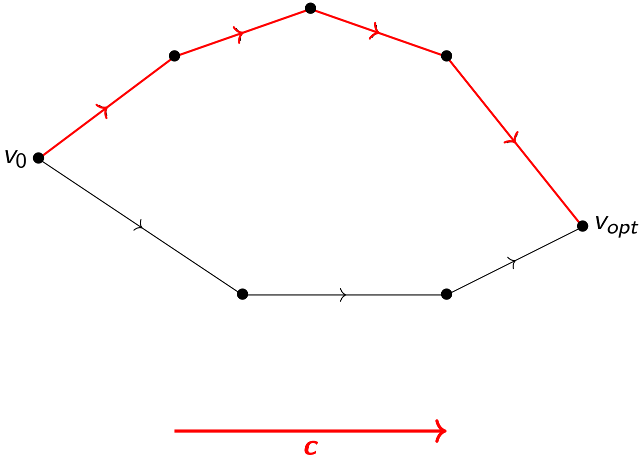


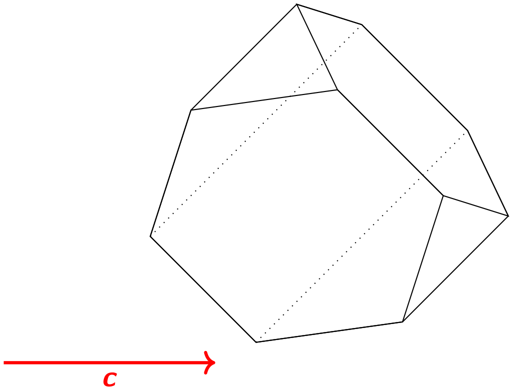
Shadow vertex rule

Optimization in dimension 2: **EASY** !



By convention, we always choose the upper path when optimizing.

Optimization in higher dimension: make it 2-dimensional !



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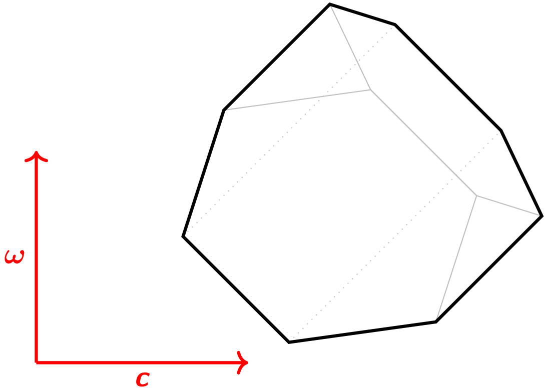

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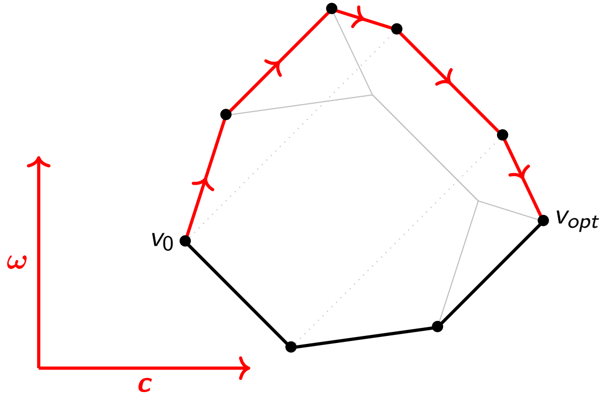

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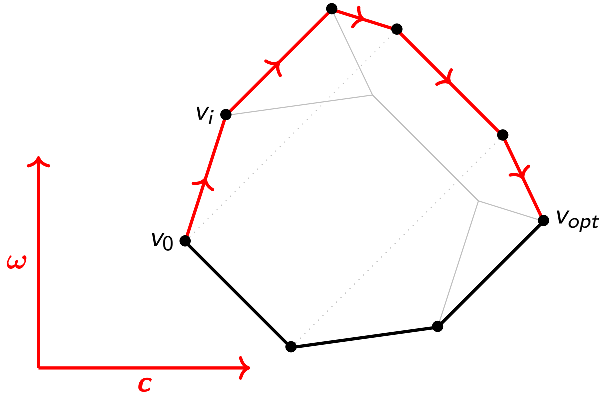

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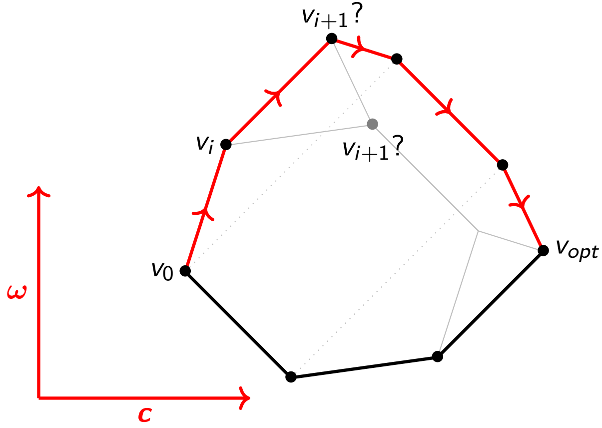

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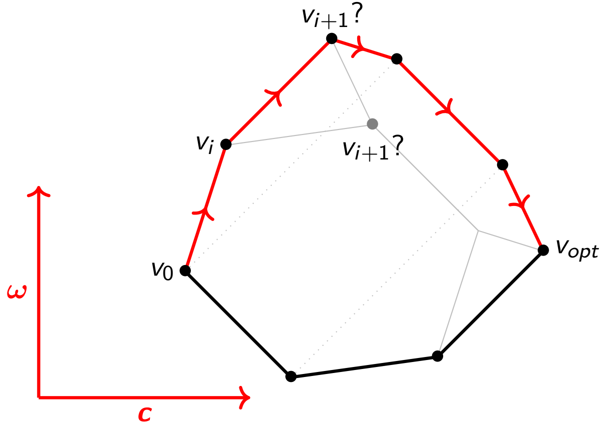
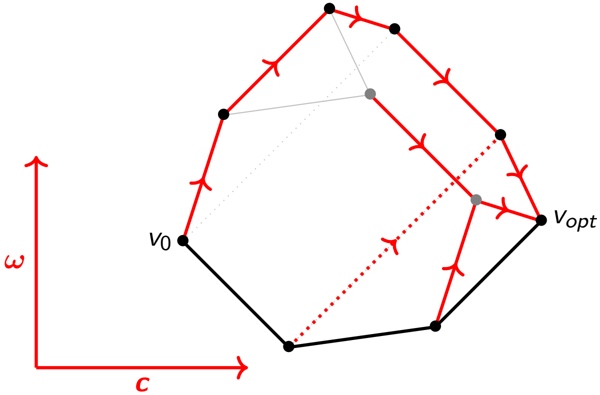


Shadow vertex rule (i.e. "take the neighbor with the best slope"):

$$A^{\omega}(v) = \operatorname{argmax}\left\{ \frac{\langle \boldsymbol{\omega}, u - v\rangle}{\langle \boldsymbol{c}, u - v\rangle}; u \text{ improving neighbor of } v \right\}$$

Shadow vertex rule

Optimization in higher dimension: make it 2-dimensional !



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Applying the rule at every vertex gives a *monotone arborescence*.

Monotone path polytope and pivot rule polytope

Let $\mathrm{P} \subset \mathbb{R}^d$ be a polytope.
Shadow vertex rule: $A^{\boldsymbol{\omega}}(v) = \operatorname{argmax}\left\{\frac{\langle \boldsymbol{\omega}, u-v\rangle}{\langle \boldsymbol{c}, u-v\rangle}; u \text{ impr. neig. of } v\right\}$.

Coherent monotone path: A monotone path that can be obtained via the shadow vertex rule.

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Monotone path polytope $\Sigma_\pi(\mathrm{P})$ [BS92]: Fiber polytope of $\mathrm{P} \xrightarrow{\pi} \mathrm{Q}$ with Q a segment. (Can be seen as a Minkowski sum of sections of P.)
The vertices of $\Sigma_\pi(\mathrm{P})$ are all coherent monotone paths.

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Coherent arborescence: An arborescence that can be obtained via the shadow vertex rule.
Pivot rule polytope $\Pi_\pi(\mathrm{P})$: Polytope which vertices are all coherent arborescences.

Monotone path polytope and pivot rule polytope

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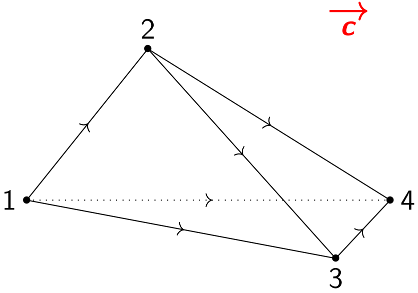
Pivot rule polytope $\Pi_\pi(\mathrm{P})$: Polytope which vertices are all coherent arborescences. (Can also be seen as a Minkowski sum of "sections".)

$$\Pi_\pi(\mathrm{P}) = \operatorname{conv}\left\{\sum_{v \neq v_{opt}} \frac{1}{\langle \boldsymbol{c}, A(v) - v\rangle}(A(v) - v); A \text{ coherent arbo. of P}\right\}$$

Case of the d-simplex



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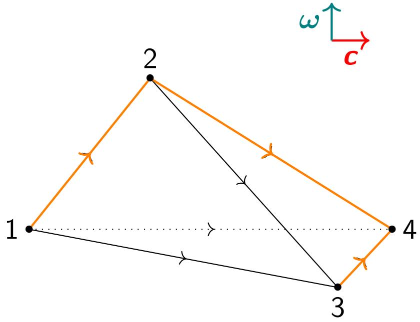

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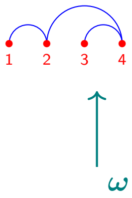
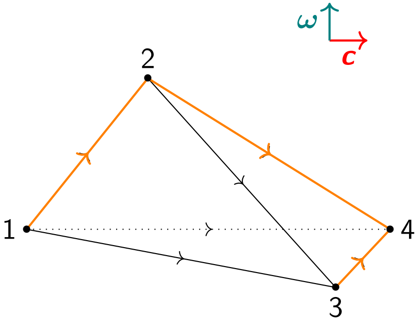
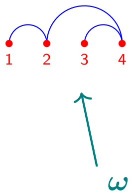
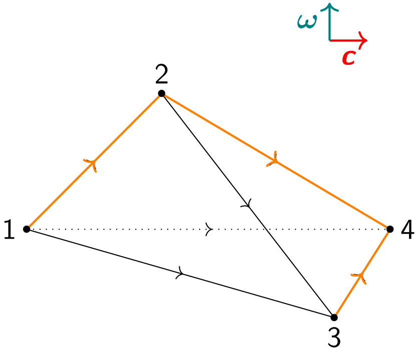
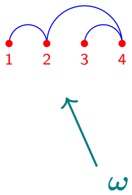


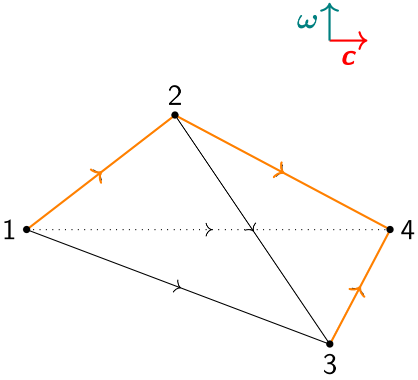



1
2
3
4
ω


ω
c
2
1
4
3

Case of the d-simplex





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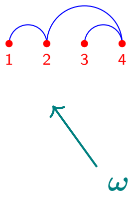

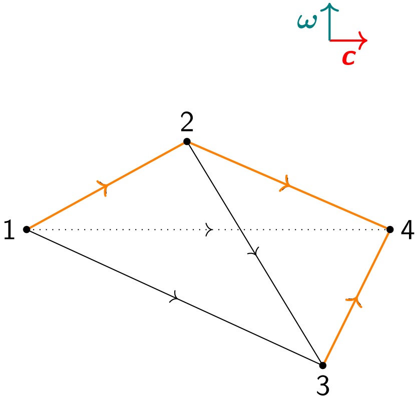

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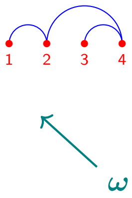

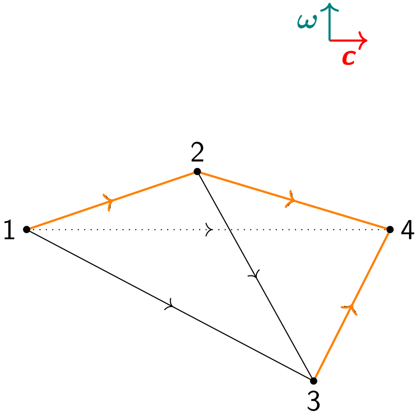

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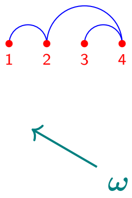

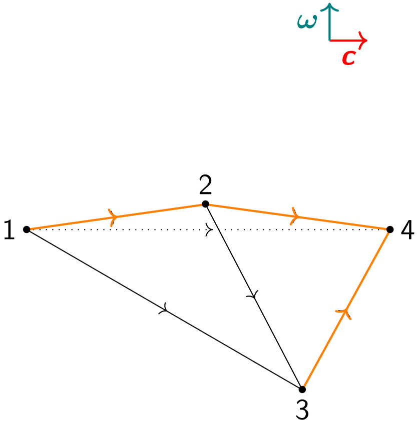

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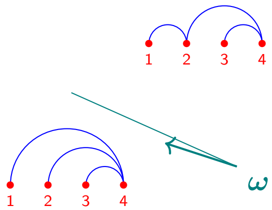

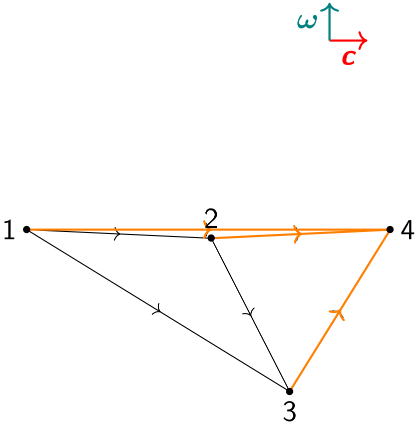

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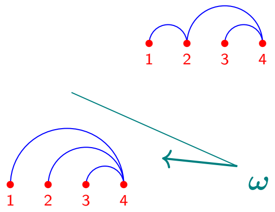

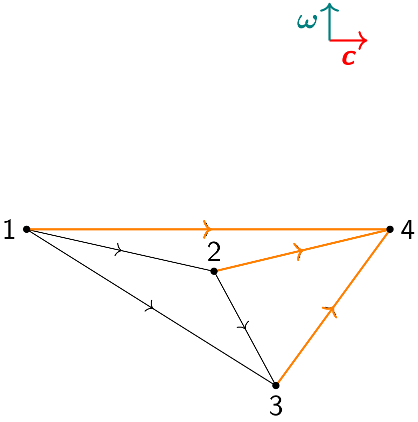

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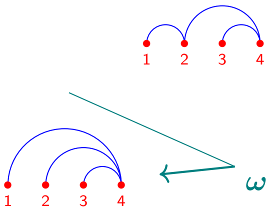

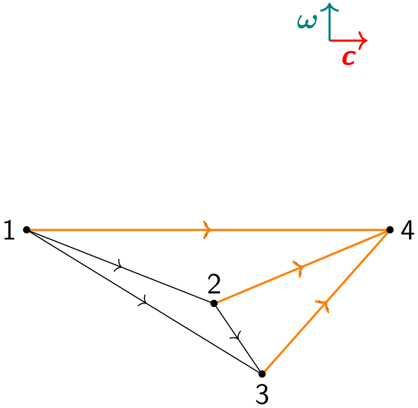

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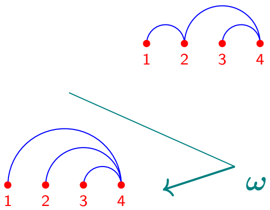

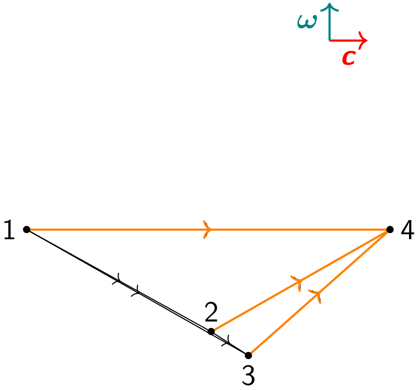

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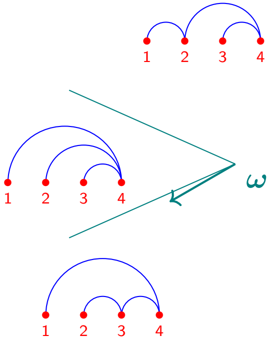

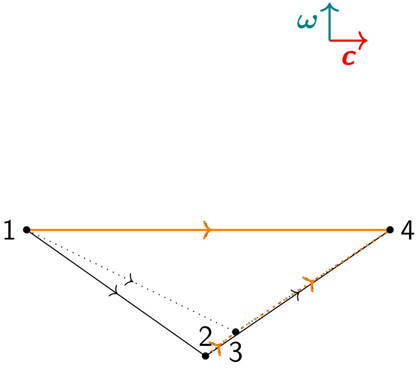

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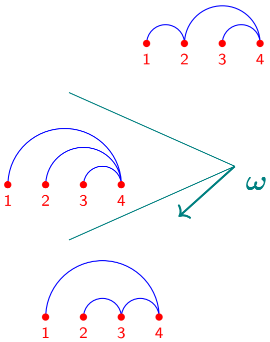

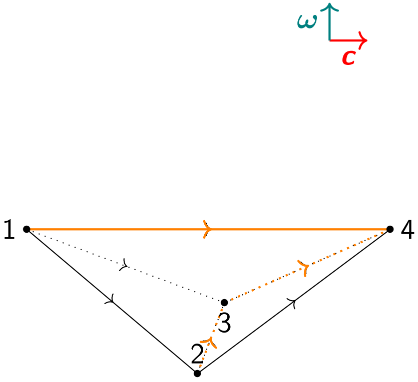

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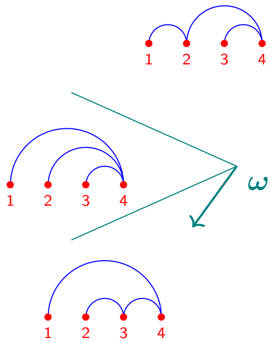

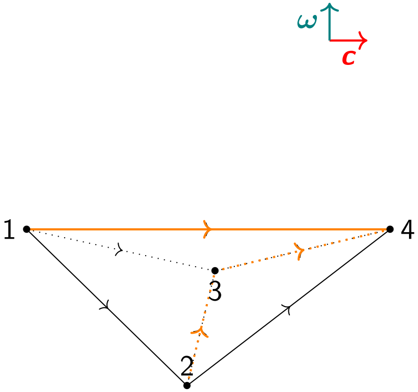

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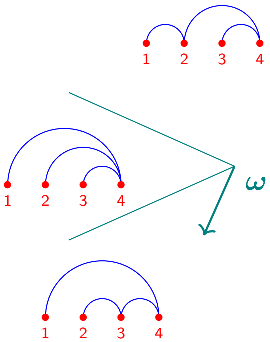

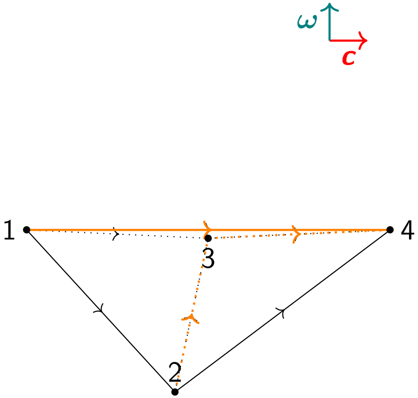

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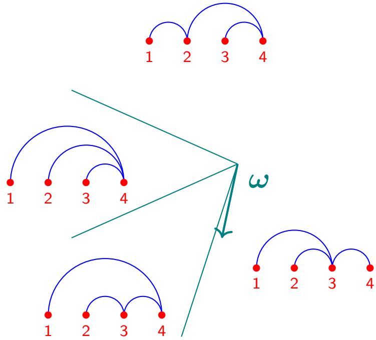

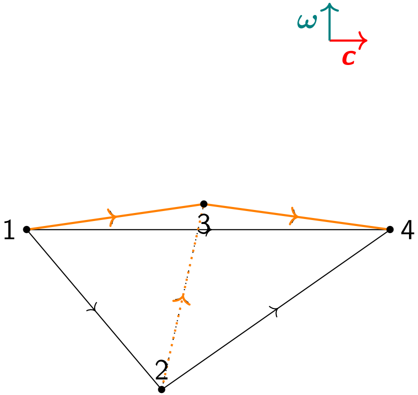

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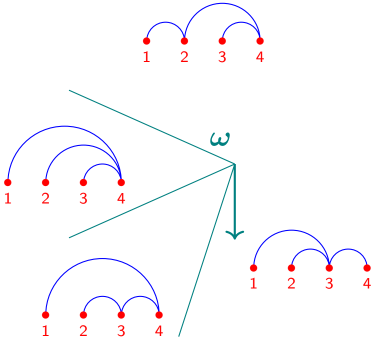

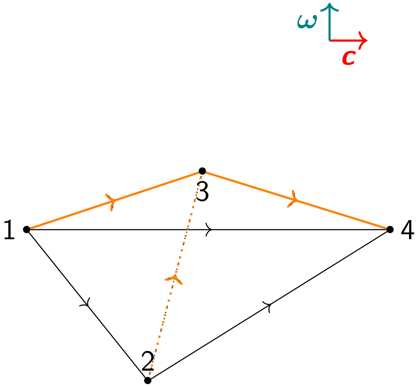

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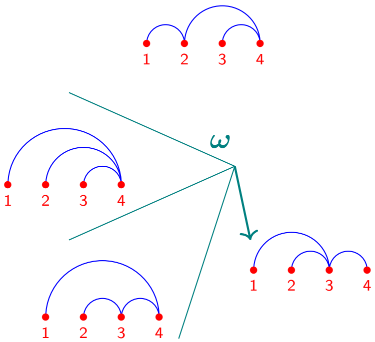

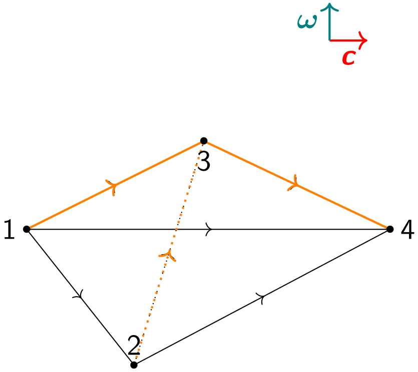

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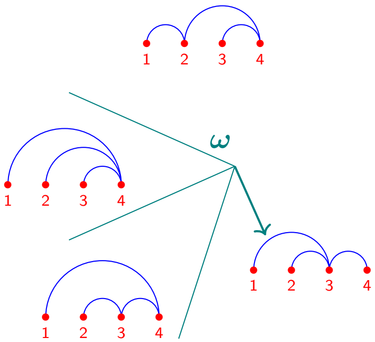

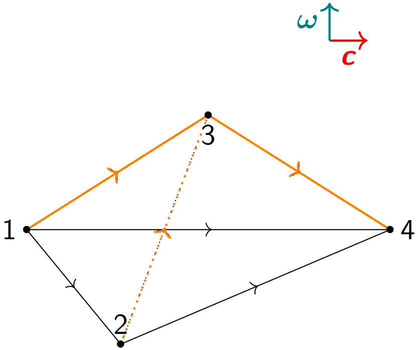

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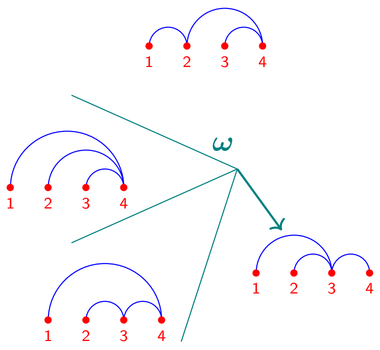

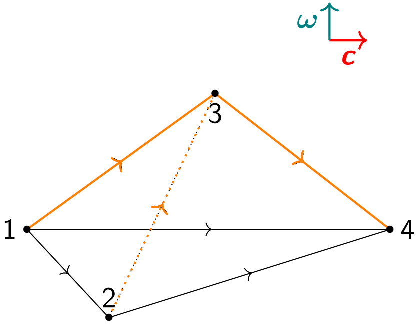

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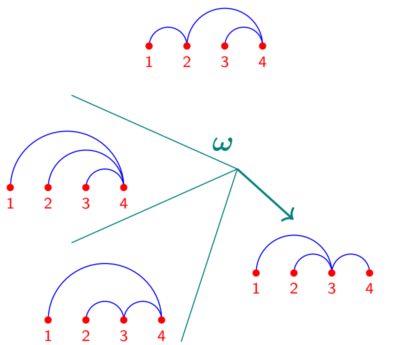

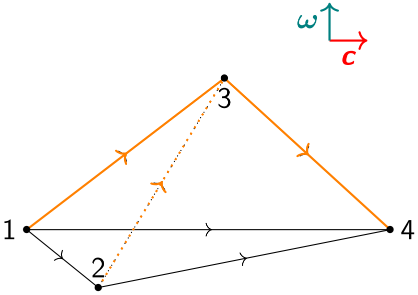

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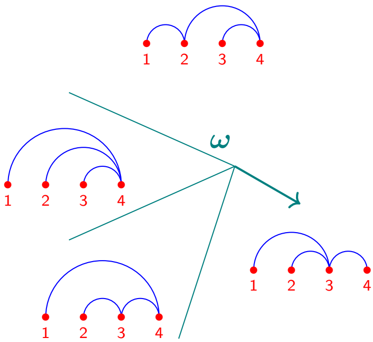

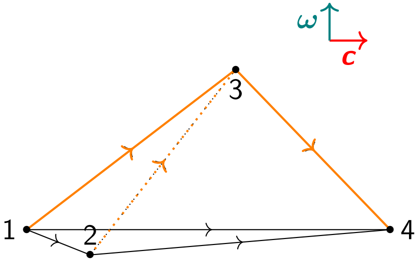

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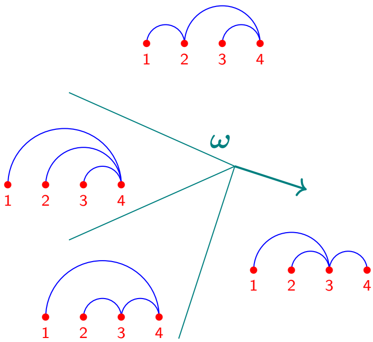

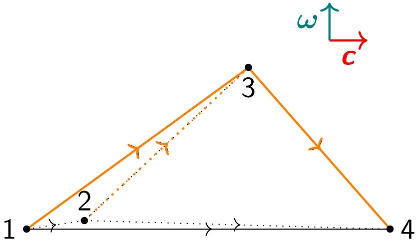

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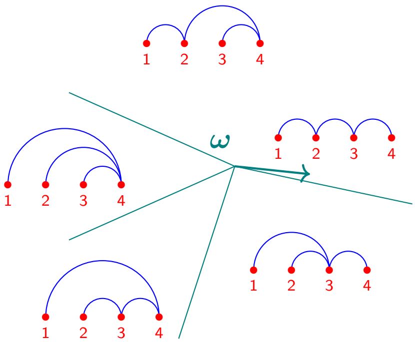

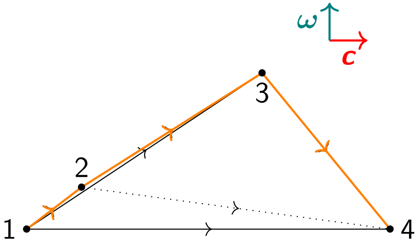

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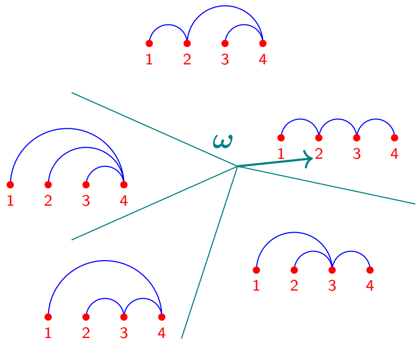

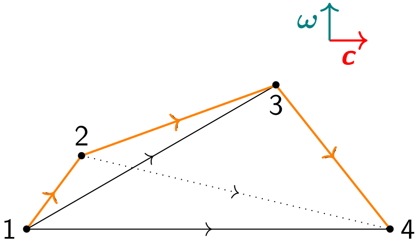

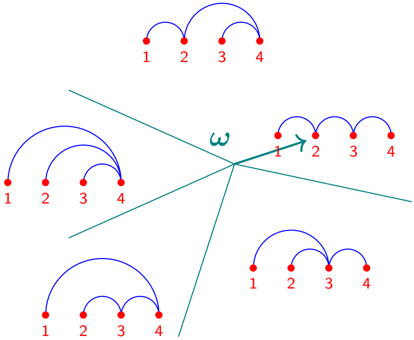
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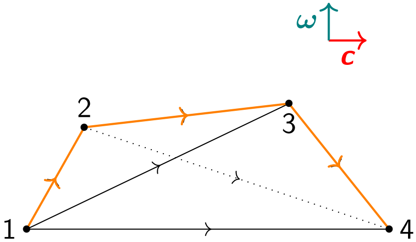
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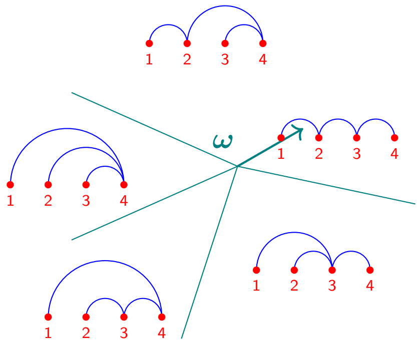
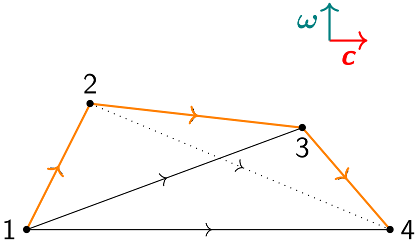
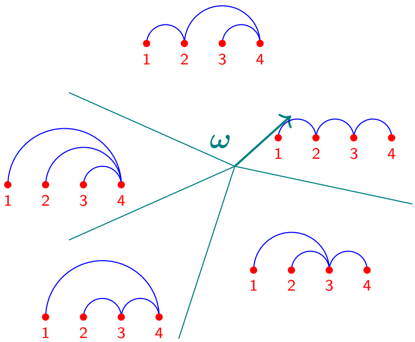
Case of the d-simplex



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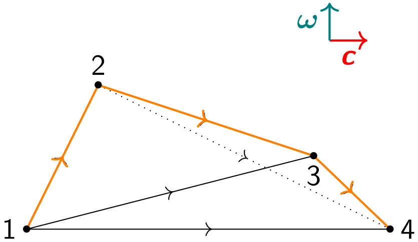
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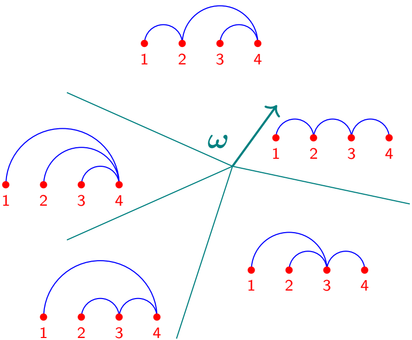

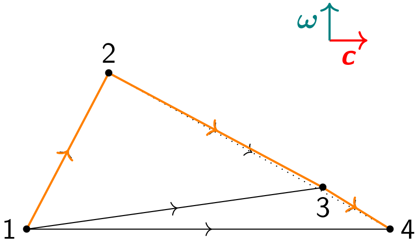

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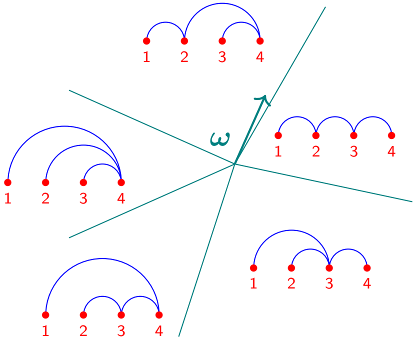

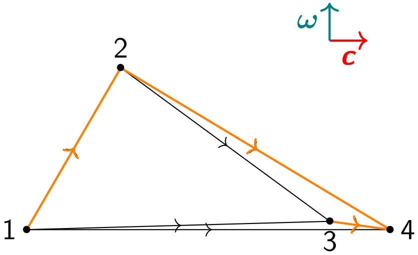

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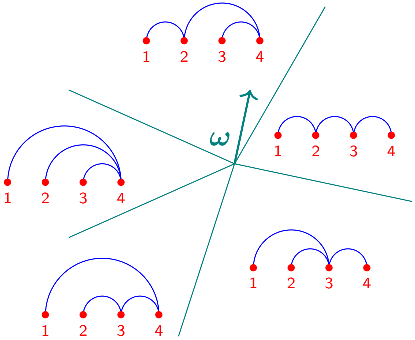

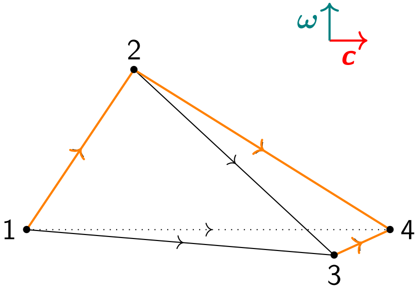

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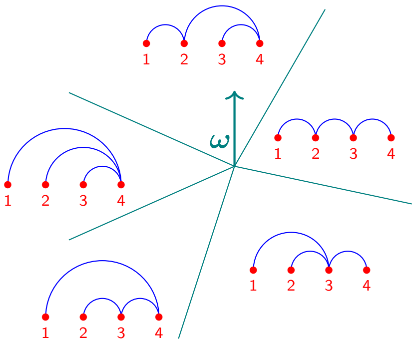

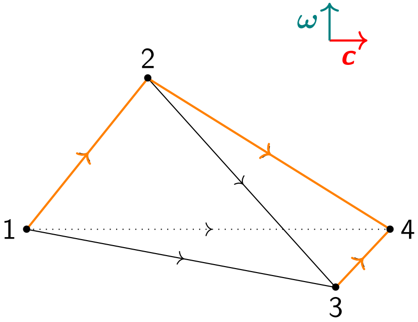

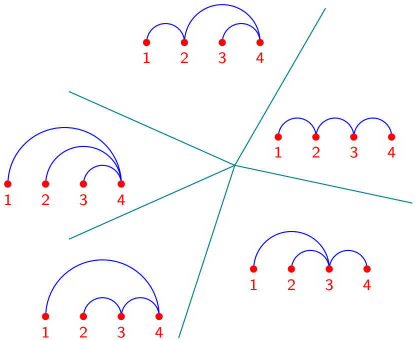

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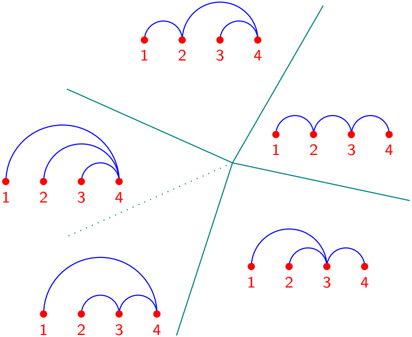
$\omega \sim \omega'$ iff $A^\omega = A^{\omega'}$.
This gives a polytopal fan [BDLLS22] (see above).



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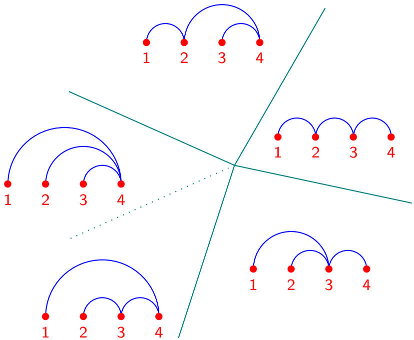

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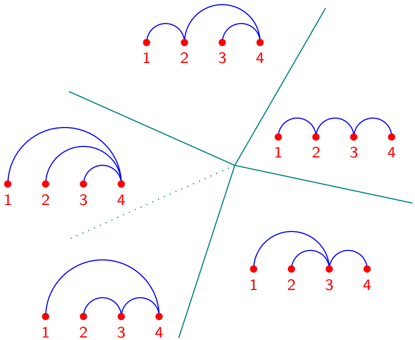
$$\begin{aligned}\Sigma_\pi(\Delta_{d+1}) &= \mathrm{Cube}_{d-1}\\ \Pi_\pi(\Delta_{d+1}) &= \mathrm{Asso}_{d-1}\end{aligned}$$



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$\Sigma_\pi(\Delta_{d+1})$ [BS92]:
A monotone path $= (v_0,$ part of the vertices, $v_{opt})$.
Choosing a monotone path = Choosing a part of the $(d-1)$-remaining vertices.
Exercise: Prove all such paths are coherent.

Case of the d-simplex

$\Pi_\pi(\Delta_{d+1})$ [BDLLSon]:

Project a simplex in dimension 2: any set of points.

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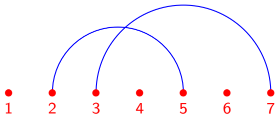

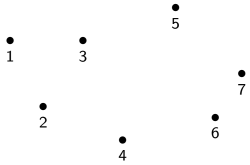

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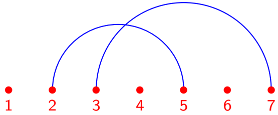

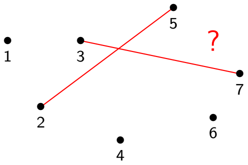

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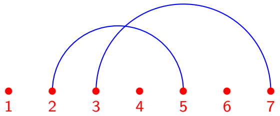

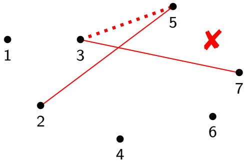

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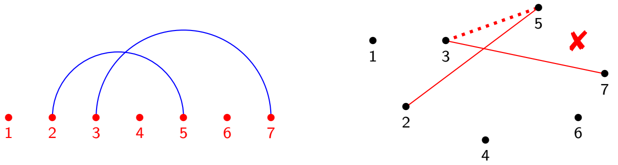
Lemma (Non-crossing)

For a polytope which graph is complete, all coherent arborescences are non-crossing.



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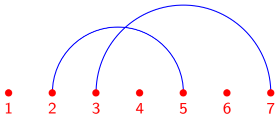
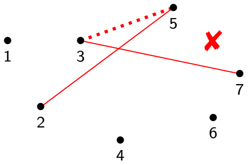
Exercise: Prove all non-crossing arborescences are coherent for the simplex.





Projection and pivot rule polytope

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Theorem (Projection and Pivot rule polytopes)

When $G_\mathsf{Q} = G_\mathsf{P}$*, then* $\Pi_\pi(\mathsf{Q}) = \rho(\Pi_{\pi\circ\rho}(\mathsf{P}))$*.*

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When $G_{\mathsf{Q}} = G_{\mathsf{P}}$, then $\Pi_\pi(\mathsf{Q}) = \rho(\Pi_{\pi\circ\rho}(\mathsf{P}))$.

Corollary (Projections of associahedra)

Pivot rule polytope of 2-neighborly polytopes are projections of associahedra.

Let's study the pivot rule polytope of cyclic polytopes!

Cyclic associahedra and intrinsic degree

Fix a dimension d and an integer $n \geq d+1$.
Cyclic polytope $\mathrm{Cyc}_d(\boldsymbol{t}) = \mathrm{conv}\,\{\gamma_d(t_1), ..., \gamma_d(t_n)\}$ where
$\gamma_d : t \mapsto (t, t^2, ..., t^d)$.
Its combinatorics **does not** depend from the choice of $t_1, ..., t_n$.
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Previous corollary: $\Pi^d_{\boldsymbol{t}}(\boldsymbol{t})$ is a (generic) projection of Asso_{d-1} (for almost all $\boldsymbol{t}$). $\Rightarrow$ Faces of $\Pi^d_{\boldsymbol{t}}(\boldsymbol{t})$ are products of associahedra.

Cyclic associahedra (Monotone path polytope)

Fix a dimension d and an integer $n \geq d + 1$.
Cyclic polytope $\mathrm{Cyc}_d(\boldsymbol{t}) = \mathrm{conv}\{\gamma_d(t_1), ..., \gamma_d(t_n)\}$ where $\gamma_d : t \mapsto (t, t^2, ..., t^d)$.

Monotone path polytope have been computed in [ADLRS00]:

$$\forall \boldsymbol{t}, \quad \Sigma_\pi(\mathrm{Cyc}_d(\boldsymbol{t})) \simeq \mathrm{Z}_{\mathrm{cyclic}}(n-2, d-1)$$

Cyclic zonotope $\mathrm{Z}_{cyclic}(n, d)$: zonotope generated by any n distinct vectors $\frac{1}{u_1}\gamma_d(u_1), ..., \frac{1}{u_n}\gamma_d(u_n)$ (**does not** depend from $u_1, ..., u_n$).

Vertices of $\Pi_{\boldsymbol{t}}^{d}(\boldsymbol{t})$ correspond to **some** non-crossing arborescences.

Cyclic associahedra

Vertices of $\Pi^d_{\boldsymbol{t}}(\boldsymbol{t})$ correspond to **some** non-crossing arborescences.

Project $\mathrm{Cyc}_d(\boldsymbol{t})$ in plane (π, ω): vertices map to $(t_i, \langle \boldsymbol{\omega} \,|\gamma_d(t_i)\rangle)$.

$$\langle \boldsymbol{\omega} \,|\gamma_d(t_i)\rangle = \sum_j \omega_j t_i^j$$

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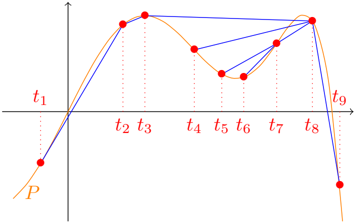

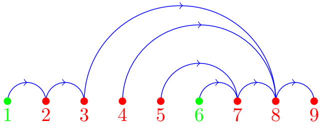

Immediate leaves of A are $\mathbb{L}(A) = \{1, 6\}$, and 6 is the unique interior immediate leaf.

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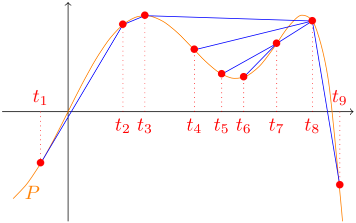

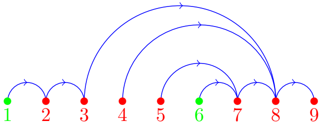

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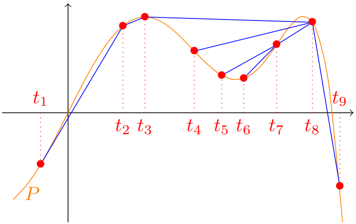
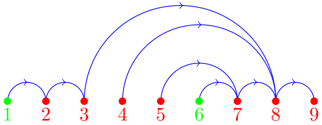
Fix a non-crossing arborescence A: can it be captured by some polynomial of degree $\leq d$?

Intrinsic degree





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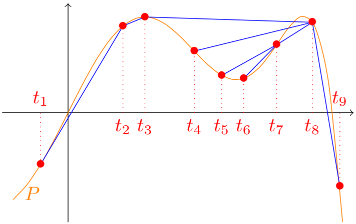
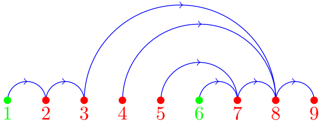
Degree $\mu(A, \boldsymbol{t}) = \min\{\deg P : A \text{ is captured by } P \text{ on } \boldsymbol{t}\}$





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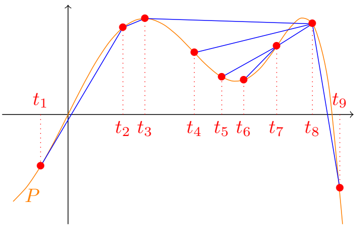

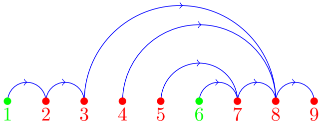

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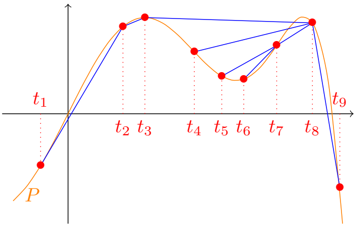
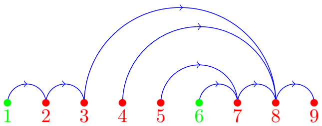
Immediate leaf i: leaf with $A(i) = i + 1$.

Theorem (Intrinsic degree)

$\mu(A) = 2\times$*(interior imm. leaves)* $+ 1\times$*(exterior imm. leaves)* $+1$

Intrinsic degree (proof)





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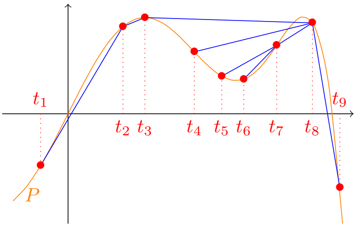
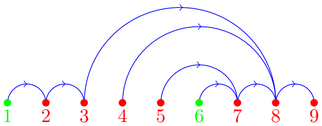
Proof that $\mu(A) \geq 2|\mathbb{L}^{\circ}(A)| + |\mathbb{L}^{\mathsf{ex}}(A)| + 1$:





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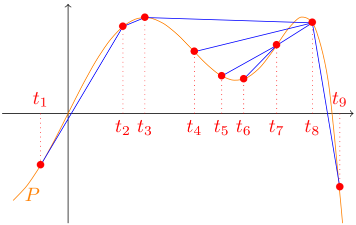

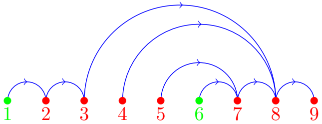

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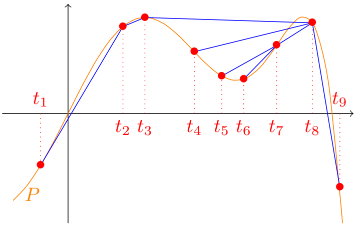

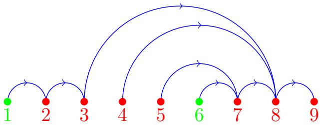

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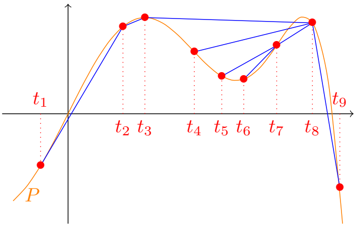

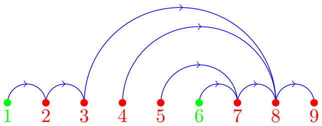

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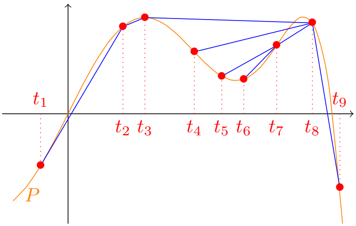
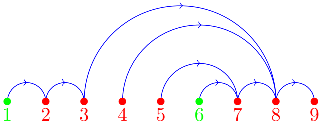
Convex and concave triangle alternate, forcing P'' to change sign.





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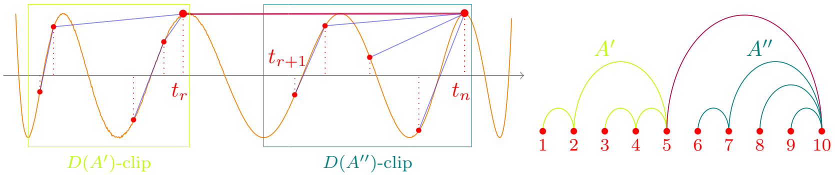
Convex and concave triangle alternate, forcing P'' to change sign.

The count of change of signs of P'' gives a minimal degree for P.



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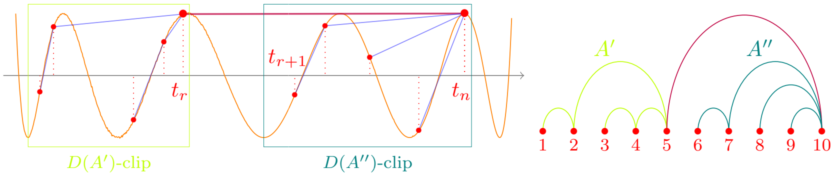

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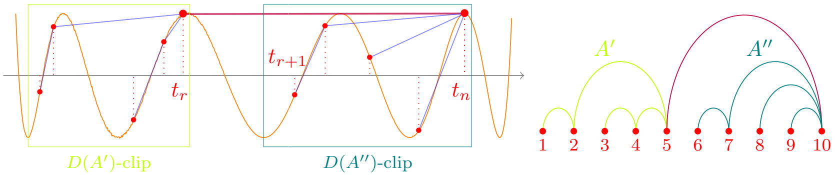
Proof by induction.

Intrinsic degree (proof)



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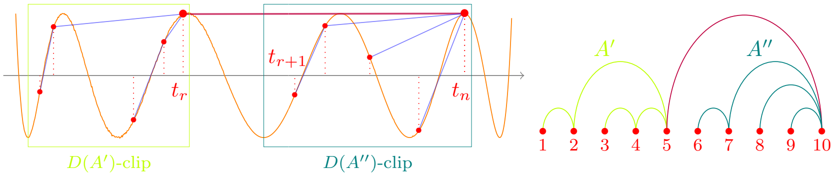

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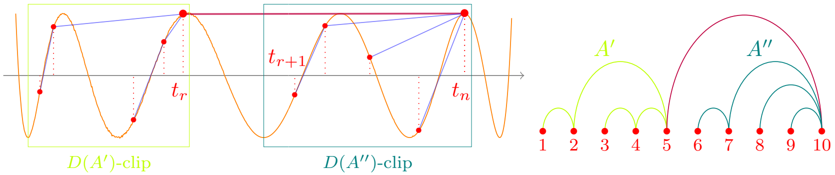
Take the Chebychev polynomial of the right degree.
Put A' on the left clip and A'' on the right clip.



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The claimed degree is made so clips fit together perfectly.

Classification of arborescences

A with $\mu(A) = 2$: exactly 1 exterior imm. leaf.

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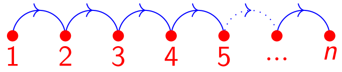
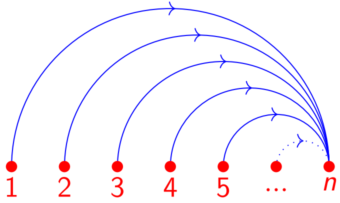


$A(i) = i + 1$



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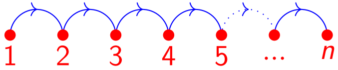

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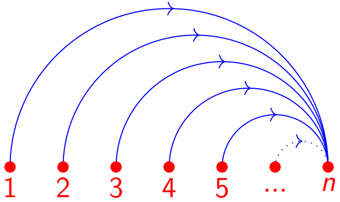

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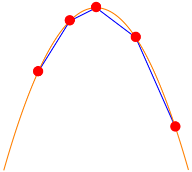

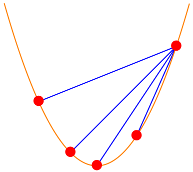

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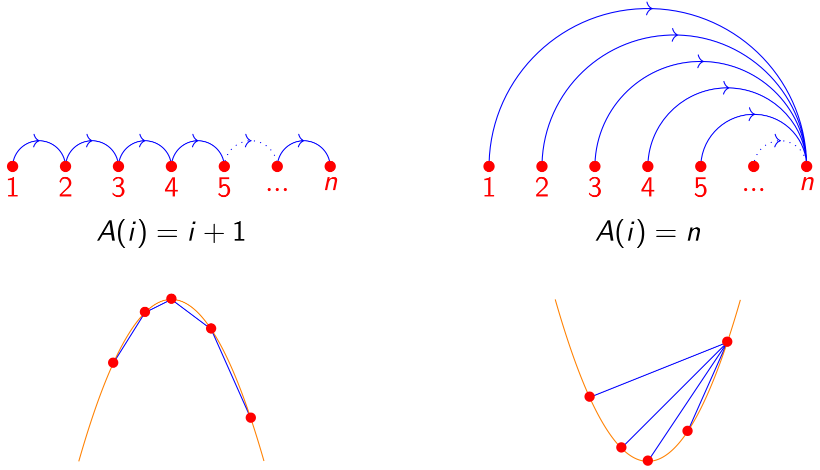

For all $\boldsymbol{t}$, $\mu(A, \boldsymbol{t}) = \mu(A) = 2$ for both quadratic arborescences A.

Classification of arborescences

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$2^{n-2} + n - 5$ such arborescences. In general: $\mu(A, \boldsymbol{t}) > \mu(A)$.

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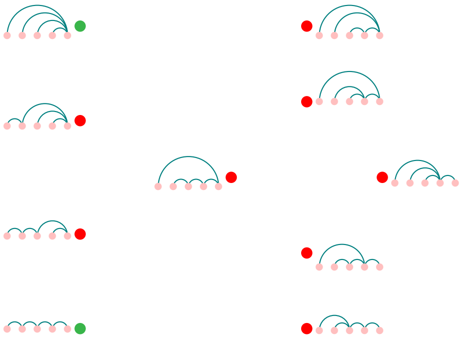

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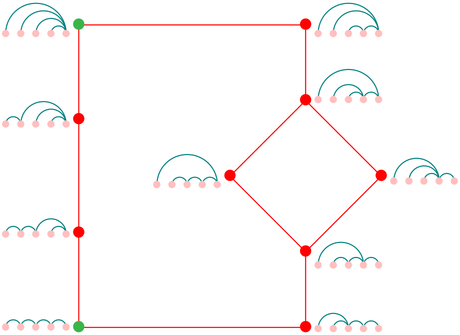

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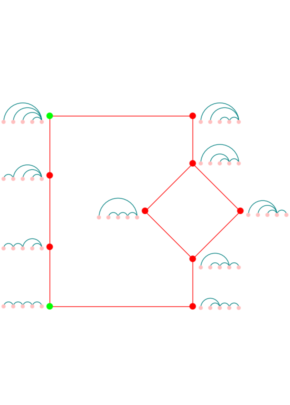

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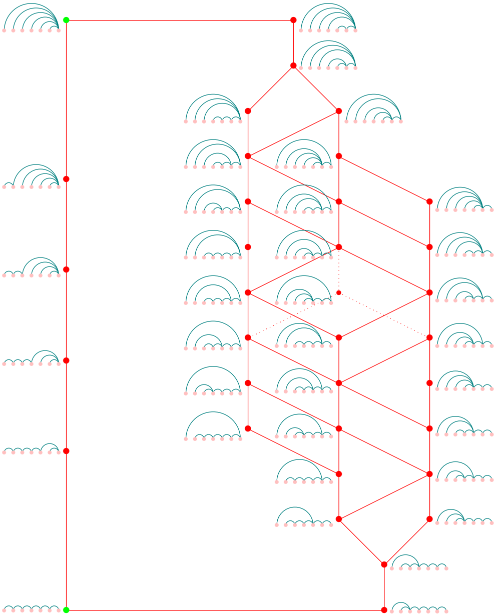

Realization sets and universal arborescences

Aim: describe the vertices of $\Pi_{\boldsymbol{t}}^{d}$.

Realization sets and universal arborescences

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Realization set $\mathcal{T}_d^{\circ}(A)$
$$\begin{aligned} &= \left\{ \boldsymbol{t} \ ; \ A \text{ is a "vertex" of } \Pi_{\boldsymbol{t}}^{d} \right\} \\ &= \{ \boldsymbol{t} \ ; \ A \text{ is captured on } \boldsymbol{t} \text{ by } P, \deg P \leq d \} \\ &= \{ \boldsymbol{t} \ ; \ \mu(A, \boldsymbol{t}) \leq d \} \end{aligned}$$

Order Cone $\mathrm{O}_n^{\circ} = \{ \boldsymbol{t} \in \mathbb{R}^n \ ; \ t_1 < \cdots < t_n \}$

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By definition (and Lagrange interpolation):

$$\mathcal{T}_1^\circ(A) \subseteq \mathcal{T}_2^\circ(A) \subseteq \cdots \subseteq \mathcal{T}_{n-1}^\circ(A) = \mathrm{O}_n^\circ$$

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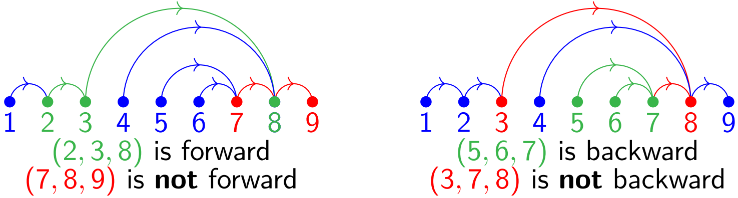
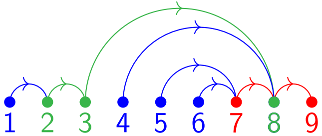
Universal arobrescence A: $\mathcal{T}_{\mu(A)}^\circ = \mathrm{O}_n^\circ$

How to describe $\mathcal{T}_d^\circ(A)$?
Who are the universal arborescences?

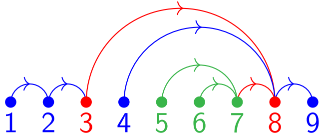
Realization sets and universal arborescences



Forward: $i \to j \to k$ with $i = \min\{v \ ; \ v \to j\}$.
Backward: $i \to k$ and $j \to k$ with $i = \max\{v < j \ ; \ v \to k\}$.



(2, 3, 8) is forward
(7, 8, 9) is **not** forward



(5, 6, 7) is backward
(3, 7, 8) is **not** backward

Forward: $i \to j \to k$ with $i = \min\{v \; ; \; v \to j\}$.
Backward: $i \to k$ and $j \to k$ with $i = \max\{v < j \; ; \; v \to k\}$.

A is captured on $\boldsymbol{t}$ by P iff:

$$\begin{cases} \forall (i,j,k) \text{ forward}, \\ \qquad (t_k - t_i)(P(t_j) - P(t_i)) - (t_j - t_i)(P(t_k) - P(t_i)) > 0 \\ \forall (a,b,c) \text{ backward}, \\ \qquad (t_c - t_a)(P(t_b) - P(t_a)) - (t_b - t_a)(P(t_c) - P(t_a)) < 0 \end{cases}$$

Proof: Look intensively at the drawing.

Farkas' trick

Note that:

$$\frac{t_k^r - t_i^r}{t_k - t_i} - \frac{t_j^r - t_i^r}{t_j - t_i}$$

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$$\frac{t_k^r - t_i^r}{t_k - t_i} - \frac{t_j^r - t_i^r}{t_j - t_i} \quad = \sum_{m+s=r-1} t_k^m t_i^s - \sum_{m+s=r-1} t_j^m t_i^s$$

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Note that:

$$
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Complete symmetric homogeneous polynomial
$h_\ell(X, Y, Z) = \sum_{p+q++s=\ell} X^p Y^q Z^s$

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A is captured on $\boldsymbol{t}$ by $P = \sum_i w_i X^i$ iff:

$$\begin{aligned} &\forall (i,j,k) \text{ forward}, && \langle h_{\cdot -2}(t_i, t_j, t_k) \,|\, \boldsymbol{w} \rangle && > 0 \\ &\forall (a,b,c) \text{ backward}, && \langle -h_{\cdot -2}(t_a, t_b, t_c) \,|\, \boldsymbol{w} \rangle && > 0 \end{aligned}$$

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Farkas' lemma: this system has a solution iff the matrix with rows $\pm h_{\cdot -2}(\boldsymbol{t})$ has no positive vector in its kernel.
Note that: $h_{-1}(\boldsymbol{t}) = 0$ and $h_0(\boldsymbol{t}) = 1$.

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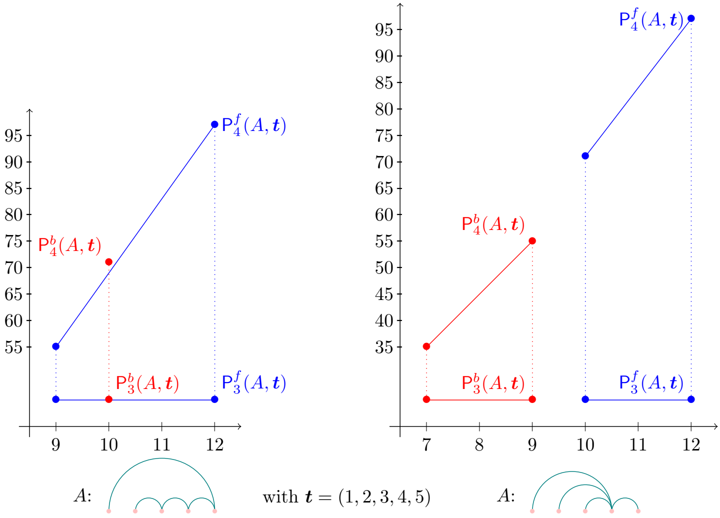
Forward and backward polytopes
$\mathrm{P}^f_d(A, \boldsymbol{t}) = \mathrm{conv}\{(h_\ell(t_i, t_j, t_k))_{1 \le \ell \le d-2} \,;\, (i,j,k) \text{ forward}\}$
$\mathrm{P}^b_d(A, \boldsymbol{t}) = \mathrm{conv}\{(h_\ell(t_a, t_b, t_c))_{1 \le \ell \le d-2} \,;\, (a,b,c) \text{ backward}\}$

Theorem (Caracterisation of $\mathcal{T}^\circ_d(A)$)

A captured on $\boldsymbol{t}$ by some P, $\deg P \le d$ iff $\mathrm{P}^f_d(A, \boldsymbol{t}) \cap \mathrm{P}^b_d(A, \boldsymbol{t}) = \emptyset$.

Farkas' trick



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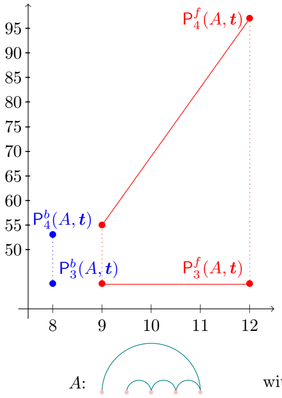
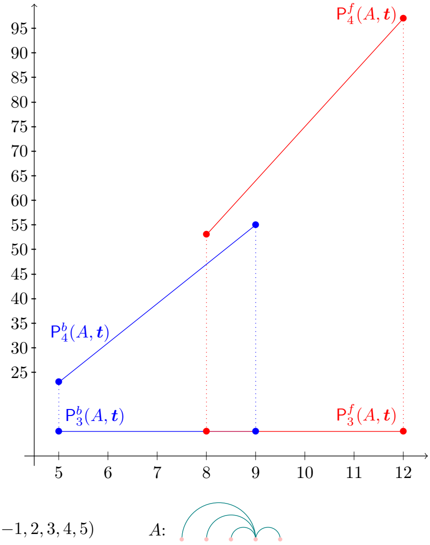


with $\boldsymbol{t} = (-1, 2, 3, 4, 5)$



Case $d \leq 3$

Carefull for $d \leq 3$, $\Pi_{\boldsymbol{t}}^{d} \neq$ projected associahedron ($G_{\mathrm{Cyc}_d(\boldsymbol{t})} \neq K_n$).

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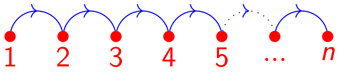

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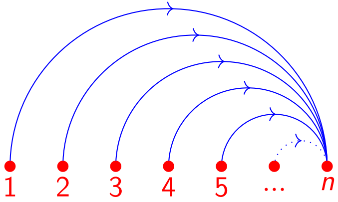
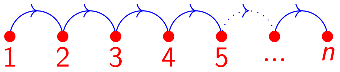


$A(i) = n$

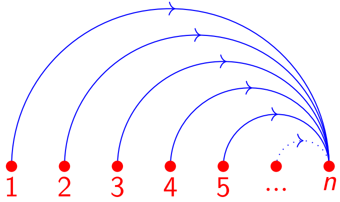
For A with $\mu(A) = 2$, either $\mathrm{P}_d^b(A, \boldsymbol{t}) = \emptyset$ or $\mathrm{P}_d^f(A, \boldsymbol{t}) = \emptyset$ (either there is no backward, or no forward).

Carefull for $d \leq 3$, $\Pi_{\boldsymbol{t}}^{d} \neq$ projected associahedron ($G_{\mathrm{Cyc}_d(\boldsymbol{t})} \neq K_n$).
But the pojected associahedron $d \leq 3$ appears as a projection of $\Pi_{\boldsymbol{t}}^{d}$ for $d \geq 4$, and gives method for $\Pi_{\boldsymbol{t}}^{2}$, $\Pi_{\boldsymbol{t}}^{3}$ (but easier to explain).

$\underline{d = 2}$:



$A(i) = i + 1$



$A(i) = n$

For A with $\mu(A) = 2$, either $\mathrm{P}_d^b(A, \boldsymbol{t}) = \emptyset$ or $\mathrm{P}_d^f(A, \boldsymbol{t}) = \emptyset$ (either there is no backward, or no forward).

Both A are universal: $\forall d \geq 2$, $\mathcal{T}_d^{\circ}(A) = \mathrm{O}_n^{\circ}$.
(but the right one is not a vertex of $\Pi_{\boldsymbol{t}}^{2}$.)

Case $d = 3$

$\underline{d = 3:}$
$\mathrm{P}^f_d(A, \boldsymbol{t}) = \mathrm{conv}\{t_i + t_j + t_k \ ; \ (i, j, k) \text{ forward}\} \subset \mathbb{R}^1$
$\mathrm{P}^b_d(A, \boldsymbol{t}) = \mathrm{conv}\{t_a + t_b + t_c \ ; \ (a, b, c) \text{ backward}\} \subset \mathbb{R}^1$

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"For which $\boldsymbol{t}$ on has $\mathrm{P}_d^f(A, \boldsymbol{t}) \cap \mathrm{P}_d^b(A, \boldsymbol{t}) = \emptyset$?": easy question!

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Minimal forward: (i, j, k) with i a leaf.
Maximal backward: (a, b, c) with $b = c - 1$.

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Theorem (Universal cubic arborescences)

There are $n + 1$ universal arborescences A with $\mu(A) = 3$ (see picture), i.e. $\mu(A) = 3$ and $\mathcal{T}^\circ_3(A) = \mathrm{O}^\circ_n$.

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Theorem (Universal cubic arborescences)

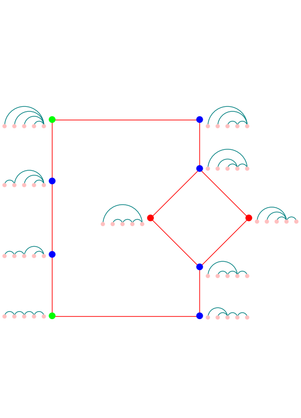
There are $n+1$ universal arborescences A with $\mu(A) = 3$ (see picture), i.e. $\mu(A) = 3$ and $\mathcal{T}_3^\circ(A) = \mathrm{O}_n^\circ$.

Theorem ((almost) facet description of $\mathcal{T}_3^\circ(A)$)

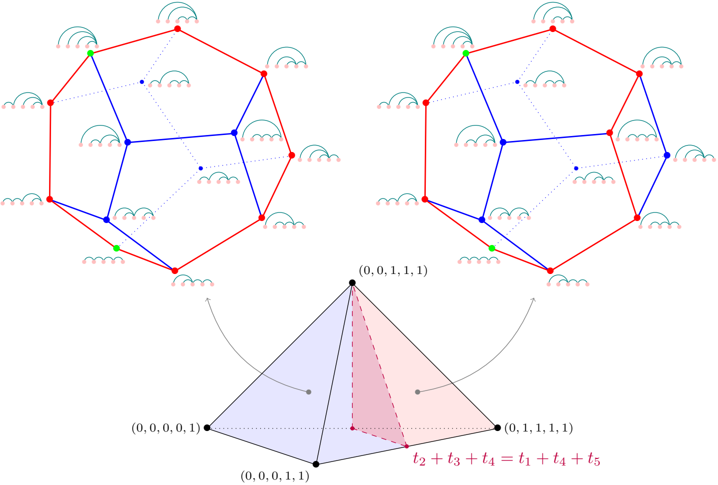
For a non-universal A with $\mu(A) = 3$:
$$\mathcal{T}_3^\circ(A) = \mathrm{O}_n^\circ \cap \{\boldsymbol{t} \ ; \ t_a + t_b + t_c < t_i + t_j + t_k \ ; \ (i,j,k) \text{ min f., and } (a,b,c) \text{ max b.}\}$$

Case $d = 3$

$\mu(A) = 2$; universal $\mu(A) = 3$; non-universal $\mu(A) = 3$



Case $d = 3$

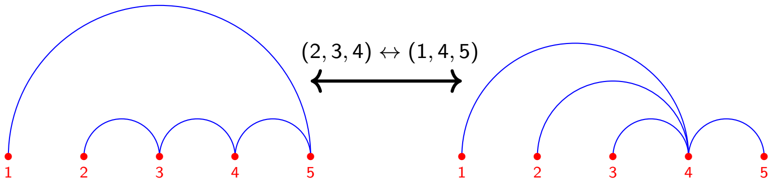


$O_5^\circ \cap \{t_1 = 0\} \cap \{t_5 = 1\}$, with the realization sets $\mathcal{T}_3^\circ(A)$.

Case $d = 3$ (double flips)

Double flip $(i, j, k) \leftrightarrow (a, b, c)$: flip the minimal forward (i, j, k) to a backward, and flip the maximal backward (a, b, c) to a forward.

Quasi-always possible to perform.



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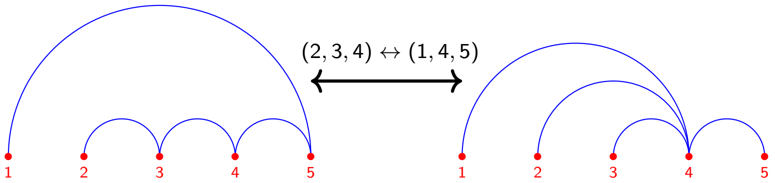

Double flip: cross some square face in the associahedron (above).

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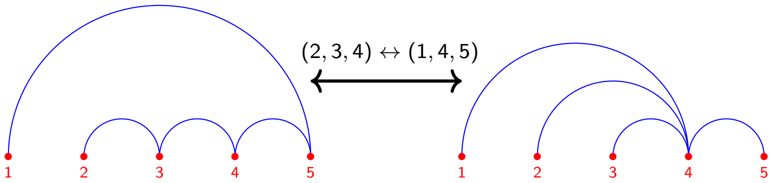

Double flip: cross some square face in the associahedron (above).
Double-flipping arrangement $\mathcal{H}_n$: arragement of hyperplans $\{t_i + t_j + t_k = t_a + t_b + t_c\}$ for (i, j, k) minimal forward and (a, b, c) maximal backward.

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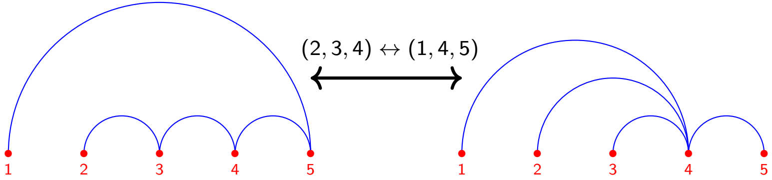

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$\mathcal{H}_n$ refines the subdivsion of O_n° induced by $\{\mathcal{T}_3^\circ(A)\}_{A;\ \mu(A)=3}$.

Case $d = 3$ (double flips)

Double-flipping arrangement $\mathcal{H}_n$: arragement of hyperplans $\{t_i + t_j + t_k = t_a + t_b + t_c\}$ for (i, j, k) min f. and (a, b, c) max b..
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For the vertices of the projected associahedron ($d = 3$):
Cross an hyperplan in $\mathcal{H}_n \Rightarrow$ loose an arobrescence **but** gain its double-flipped.

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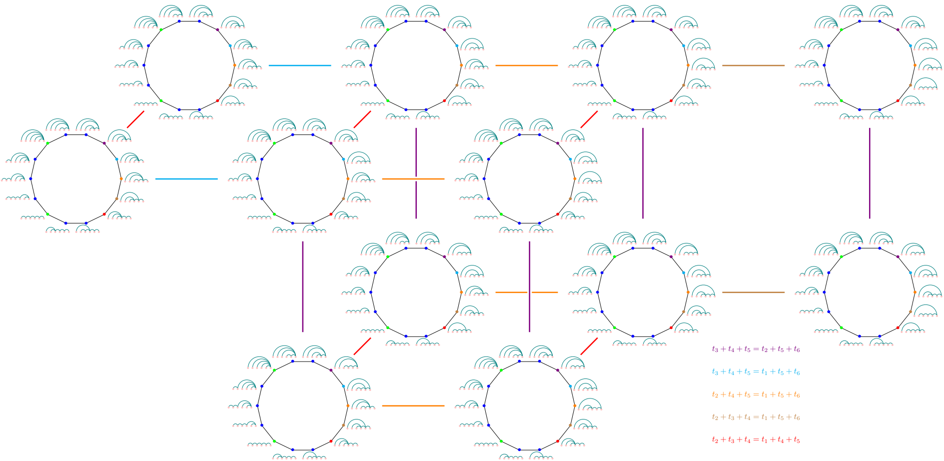
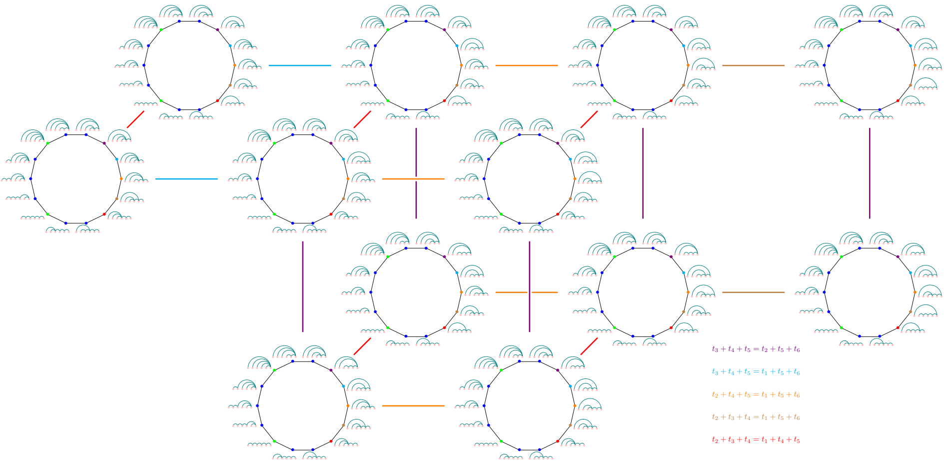
For the vertices of the projected associahedron ($d = 3$):
Cross an hyperplan in $\mathcal{H}_n$ $\Rightarrow$ loose an arobrescence **but** gain its double-flipped.

Corollary (Number of vertices of the 3-projected associahedron)

The number of vertices of the projected associahedron for $d = 3$ does not depend on $\boldsymbol{t}$, namely it is $\binom{n}{2} - 1$.

Case $d = 3$ (double flips)





Thank you!

For $d \geq 4$, everything gets harder...

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